

# CALCULATION OF UNIL FOR THE CYCLIC GROUP OF ORDER TWO

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ABSTRACT. Cappell's unitary nilpotent groups  $\mathrm{UNil}_*^h(R; R, R)$  are calculated for the integral group ring  $R = \mathbb{Z}[C_2]$  of the cyclic group  $C_2$  of order two. Specifically, they are determined as modules over the Verschiebung algebra  $\mathcal{V}$  using the Connolly–Ranicki isomorphism [CR05] and the Connolly–Davis relations [CD04].

## 1. INTRODUCTION

Consider the simplest nontrivial 2-group

$$C_2 = \langle T \mid T^2 = 1 \rangle.$$

Observe that the integral group ring  $\mathbb{Z}[C_2]$  fits into Rim's cartesian square

$$\begin{array}{ccc} \mathbb{Z}[C_2] & \xrightarrow{i^-} & \mathbb{Z} \\ i^+ \downarrow & & j^- \downarrow \\ \mathbb{Z} & \xrightarrow{j^+} & \mathbb{F}_2 \end{array}$$

of rings with involution, where  $i^\pm(T) = \pm 1$ . We focus on piecing together its UNil as a module from the UNil of the component rings  $\mathbb{Z}$  and  $\mathbb{F}_2$ , using a Mayer–Vietoris sequence in algebraic  $L$ -theory. The additional structure<sup>1</sup> on

$$\mathrm{UNil}_*^h(R) := \mathrm{UNil}_*^h(R; R, R) \cong NL_*^h(R)$$

computed below is its covariant (pushforward) module structure over the **Verschiebung algebra**

$$\mathcal{V} := \mathbb{Z}[V_n \mid n > 0] = \mathbb{Z}[V_p \mid p \text{ prime}]$$

of  $n$ -th power operators

$$V_n := (x \mapsto x^n)$$

on polynomial rings  $R[x]$ . An analogous structure in algebraic  $K$ -theory has been studied by Joachim Grunewald [Gru, §4.2.2] for the Bass Nil-groups

$$\widetilde{\mathrm{Nil}}_*(R) = NK_{*+1}(R).$$

The following two theorems are the main results of this paper. In the instance  $F = C_2$ , since the lower  $\widetilde{\mathrm{Nil}}_i$ - and  $NK_{i+1}$ -groups vanish [Har87], we may replace the  $\langle -\infty \rangle$  decoration with the  $h$  decoration. The first main theorem provides a general vanishing result and a classifying isomorphism, specializing [Kha].

<sup>1</sup>The  $\mathcal{V}$ -module structure is induced by the Connolly–Ranicki isomorphism (Thm. 2.2).

**Theorem 1.1.** *Suppose  $F$  is a finite group that contains a normal Sylow 2-subgroup of exponent two. If  $n \equiv 0, 1 \pmod{4}$ , then the following abelian group vanishes:*

$$\mathrm{UNil}_n^{\langle -\infty \rangle}(\mathbb{Z}[F]) = 0.$$

*Furthermore, if  $n \equiv 2 \pmod{4}$ , then the following induced map is an isomorphism:*

$$\mathrm{UNil}_n^{\langle -\infty \rangle}(\mathbb{Z}[F]) \longrightarrow \mathrm{UNil}_n^{\langle -\infty \rangle}(\mathbb{F}_2) \xrightarrow{r \cong} NL_n^{\langle -\infty \rangle}(\mathbb{F}_2) \xrightarrow{\mathrm{Arf} \cong} x\mathbb{F}_2[x]/(f^2 - f).$$

The second main theorem examines non-vanishing in the remaining dimensions.

**Theorem 1.2.** *If  $n \equiv 3 \pmod{4}$ , then there exists a decomposition*

$$\mathrm{UNil}_n^{\langle -\infty \rangle}(\mathbb{Z}[C_2]) \cong \mathrm{UNil}_{n+1}^{\langle -\infty \rangle}(\mathbb{F}_2) \oplus \mathrm{UNil}_n^{\langle -\infty \rangle}(\mathbb{Z}) \oplus \mathrm{UNil}_n^{\langle -\infty \rangle}(\mathbb{Z}).$$

*Proof.* Immediate from Theorems 2.4 and 2.11.  $\square$

## 2. DEFINITIONS, RELATIONS, AND DECOMPOSITIONS

Unless specified otherwise, all the surgery groups  $L, \mathrm{UNil}, NL$  in this paper shall have the  $h$  decoration with respect to the algebraic  $K$ -groups  $\tilde{K}_1, \tilde{\mathrm{Nil}}_0, NK_1$ .

**Definition 2.1** (Bass). Let  $R$  be a ring with involution. For each  $n \in \mathbb{Z}$ , define the abelian group

$$NL_n(R) := \mathrm{Ker}(\mathrm{aug}_0 : L_n(R[x]) \longrightarrow L_n(R)).$$

Therefore there is a natural decomposition

$$L_n(R[x]) = L_n(R) \oplus NL_n(R).$$

The subsequent statements are technical tools for the main theorems. The first is the Connolly–Ranicki isomorphism [CR05, Thm. A], which is a fundamental equivalence in the computation of a certain class of  $\mathrm{UNil}$ -groups.

**Theorem 2.2** (Connolly–Ranicki). *Let  $R$  be a ring with involution. Then for all  $n \in \mathbb{Z}$ , there is a natural isomorphism*

$$r^h : \mathrm{UNil}_n^h(R; R, R) \longrightarrow NL_n^h(R)$$

*which descends to a natural isomorphism*

$$r^{\langle -\infty \rangle} : \mathrm{UNil}_n^{\langle -\infty \rangle}(R; R, R) \longrightarrow NL_n^{\langle -\infty \rangle}(R).$$

**Remark 2.3.** According to Connolly–Koźniewski [CK95], Connolly–Ranicki [CR05], and Connolly–Davis [CD04], the group  $NL_{\mathrm{odd}}(\mathbb{F}_2)$  vanishes, and the Arf invariant is an isomorphism:

$$\mathrm{Arf} : NL_{\mathrm{even}}(\mathbb{F}_2) \longrightarrow \frac{x\mathbb{F}_2[x]}{(f^2 - f)}.$$

The inverse of Arf is given by the map  $q \mapsto P_{q,1}$ , where for all  $p, g \in \mathbb{Z}[x]$  the symplectic form  $P_{p,g}$  is defined by

$$P_{p,g} := \left( \bigoplus_2 \mathbb{F}_2[x], \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} p \\ g \end{pmatrix} \right).$$

Also the group  $NL_n(\mathbb{Z})$  vanishes if  $n \equiv 0, 1 \pmod{4}$ , the induced map to  $NL_2(\mathbb{F}_2)$  is an isomorphism if  $n \equiv 2 \pmod{4}$ , and there is a two-stage obstruction theory [CD04, Proof 1.7] if  $n \equiv 3 \pmod{4}$ :

$$0 \longrightarrow \frac{x\mathbb{F}_2[x]}{(f^2 - f)} \xrightarrow{\mathcal{P}} NL_3(\mathbb{Z}) \xrightarrow{B} x\mathbb{F}_2[x] \times x\mathbb{F}_2[x] \longrightarrow 0.$$

It is given primarily by certain characteristic numbers  $B$  in Wu classes of  $(-1)$ -quadratic linking forms over  $(\mathbb{Z}[x], 2)$ , and secondarily by the Arf invariant, of even linking forms  $\mathcal{P}$ , over the function field  $\mathbb{F}_2(x)$ .

**Theorem 2.4.** *Consider  $P = C_2$  with trivial orientation character. Then, as Verschiebung modules, there is a decomposition*

$$NL_3(\mathbb{Z}[C_2]) = NL_3(\mathbb{Z}) \oplus \widetilde{NL}_3(\mathbb{Z}[C_2])$$

and there is an exact sequence (constituting a three-stage obstruction theory):

$$0 \longrightarrow NL_0(\mathbb{F}_2) \xrightarrow{\tilde{\partial}} \widetilde{NL}_3(\mathbb{Z}[C_2]) \xrightarrow{i^-} NL_3(\mathbb{Z}) \longrightarrow 0.$$

Ingredients for the next theorem are as follows. Given a ring  $A$  with involution and  $\epsilon = \pm 1$ , there is an identification [Ran81, Prop. 1.6.4] between split  $\epsilon$ -quadratic formations over  $A$  and connected 1-dimensional  $\epsilon$ -quadratic complexes over  $A$ . The identification between  $(-\epsilon)$ -quadratic linking forms over  $(A, (2)^\infty)$  and resolutions by  $(2)^\infty$ -acyclic 1-dimensional  $\epsilon$ -quadratic complexes over  $A$  is given by [Ran81, Proposition 3.4.1].

The determination of the above extension (2.4) of abelian groups involves algebraic gluing of quadratic complexes [Ran81, §1.7], given below (2.7) by a choice  $\mathcal{M}$  of set-wise section. Recall from group cohomology that an extension of abelian groups

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

and a choice of set-wise section  $s : C \rightarrow B$  determine a **factorset**

$$f : C \times C \longrightarrow A; \quad (c, c') \mapsto s(c) + s(c') - s(c + c').$$

Our main concern is the computation of such a function  $f$ , via generators of  $C$  and an invariant for  $A$  in the above sequence (2.4) of abelian groups.

*Remark 2.5.* The Connolly–Davis computation of  $NL_3(\mathbb{Z}) \cong NL_4(\mathbb{Z}, (2)^\infty)$  involves generators  $\mathcal{N}_{p,g}$  indexed by polynomials  $p, g \in \mathbb{Z}[x]$ . Either  $p$  or  $g$  must have zero constant coefficient, and each generator is defined as the nonsingular  $(+1)$ -quadratic linking form

$$\mathcal{N}_{p,g} := \left( \bigoplus_2 \mathbb{Z}[x]/2, \begin{pmatrix} p/2 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \begin{pmatrix} p/2 \\ g \end{pmatrix} \right)$$

of exponent two over  $(\mathbb{Z}[x], (2)^\infty)$ , see [CD04, Dfn. 1.6 and p. 1057]. For our computation, we identify it with a choice of resolution by a nonsingular split  $(-1)$ -quadratic formation

$$\mathcal{N}_{p,g} = \left( \bigoplus_2 \mathbb{Z}[x], \left( \begin{pmatrix} p & 1 \\ 1 & 2g \\ 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} p & 1 \\ 1 & 2g \end{pmatrix} \right) \bigoplus_2 \mathbb{Z}[x] \right).$$

**Definition 2.6** ([Ran81, p. 69]). Let  $R$  be a ring with involution, and let  $F, G$  be finitely generated projective  $R$ -modules. A **nonsingular split  $\epsilon$ -quadratic formation**  $\left( F, \left( \begin{pmatrix} \gamma \\ \mu \end{pmatrix}, \theta \right) G \right)$  **over**  $R$  consists of the hyperbolic  $\epsilon$ -quadratic form

$$\mathcal{H}_\epsilon(F) := \left( F \oplus F^*, \begin{pmatrix} 0 & \mathbf{1}_F \\ 0 & 0 \end{pmatrix} \right)$$

along with the standard lagrangian  $F \oplus 0$ , a second lagrangian

$$\mathrm{Im}\left(\begin{pmatrix} \gamma \\ \mu \end{pmatrix} : G \rightarrow F \oplus F^*\right),$$

and a hessian  $\theta : G \rightarrow G^*$ , which is a de-symmetrization of the pullback form:

$$\theta - \epsilon\theta^* = \begin{pmatrix} \gamma \\ \mu \end{pmatrix}^* \begin{pmatrix} 0 & \mathbf{1}_F \\ 0 & 0 \end{pmatrix} = \gamma^* \circ \mu : G \longrightarrow G^*.$$

□

**Definition 2.7.** For any polynomial  $q \in x\mathbb{Z}[x]$ , define the nonsingular split  $(-1)$ -quadratic formation  $\mathcal{Q}_q$  over  $\mathbb{Z}[C_2][x]$ , where  $\hat{q} := 2(1-T)q$ , by

$$\mathcal{Q}_q := \left( \bigoplus_2 \mathbb{Z}[C_2][x], \left( \begin{pmatrix} 0 & \hat{q} \\ \hat{q} & 0 \\ 1 & (1-T)q \\ (1-T) & 1 \end{pmatrix}, \begin{pmatrix} \hat{q} & 0 \\ \hat{q} & q\hat{q} \end{pmatrix} \right) \bigoplus_2 \mathbb{Z}[C_2][x] \right).$$

For any polynomials  $p, g \in \mathbb{Z}[x]$  with  $pg \in x\mathbb{Z}[x]$ , define the nonsingular split  $(-1)$ -quadratic formation  $\mathcal{M}_{p,g}$  over  $\mathbb{Z}[C_2][x]$  by

$$\mathcal{M}_{p,g} := \left( \bigoplus_2 \mathbb{Z}[C_2][x], \left( \begin{pmatrix} p & 1 \\ 1 & (1-T)g \\ 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} p & 1 \\ 1 & (1-T)g \end{pmatrix} \right) \bigoplus_2 \mathbb{Z}[C_2][x] \right).$$

□

Indeed each of these  $(-1)$ -quadratic formations consists of lagrangian summands, since the associated 1-dimensional  $(-1)$ -quadratic complex over  $\mathbb{Z}[C_2][x]$  is connected [Ran80, Proof 2.3] and in fact Poincaré: the Poincaré duality map on the level of projective modules induces isomorphisms on the homology groups. For example in  $\mathcal{M}_{p,g}$ , the nontrivial homological Poincaré duality map is

$$\begin{pmatrix} p & 1 \\ 1 & (1-T)g \end{pmatrix} : H^0(C) \rightarrow H_1(C), \quad \text{where } H^0(C) = H_1(C) = \mathbb{Z}[C_2][x]/2.$$

Its determinant  $(1-T)pg - 1$  is a unit mod 2 in the commutative ring  $\mathbb{Z}[C_2][x]$ , since

$$((1-T)pg - 1)^2 = 2(1-T)(pg)^2 - 2(1-T)pg + 1 \equiv 1 \pmod{2}.$$

Therefore the Poincaré duality map for  $\mathcal{M}_{p,g}$  is a homology isomorphism. Also, the formation  $\mathcal{Q}_q$  is obtained as a pullback of a nonsingular formation, cf. Proof 2.8(1).

**Proposition 2.8.** *The following formulas are satisfied for cobordism classes in the reduced module  $\widetilde{NL}_3(\mathbb{Z}[C_2])$ .*

- (1) *Boundary map:*  $\widetilde{\partial}[P_{q,1}] = [\mathcal{Q}_q]$
- (2) *Lifts:*  $i^-[\mathcal{M}_{p,g}] = [\mathcal{N}_{p,g}]$  and  $i^+[\mathcal{M}_{p,g}] = 0$

Now we state the basic relations between our generators  $\mathcal{Q}$  and  $\mathcal{M}$ , established by algebraic surgery. Their inspiration is the statement and proof of [CD04, Lemma 4.3], but they are proven independently.

**Proposition 2.9.** *The following formulas are satisfied for cobordism classes in the reduced module  $\widetilde{NL}_3(\mathbb{Z}[C_2])$ .*

- (1) *Additivity*:  $[\mathcal{M}_{p_1,g}] + [\mathcal{M}_{p_2,g}] = [\mathcal{M}_{p_1+p_2,g}] + [\mathcal{Q}_q]$  where  $q := (p_1g)(p_2g)$
- (2) *Symmetry*:  $[\mathcal{M}_{2p,g}] = [\mathcal{M}_{2g,p}]$
- (3) *Square associativity*:  $[\mathcal{M}_{x^2p,g}] = [\mathcal{M}_{p,x^2g}]$
- (4) *Square root*:  $[\mathcal{M}_{2p^2g,g}] = [\mathcal{M}_{2p,g}]$

Here are some useful formal consequences, which do not require the technique of algebraic surgery.

**Corollary 2.10.** *The following formulas are satisfied for cobordism classes in the reduced module  $\widetilde{NL}_3(\mathbb{Z}[C_2])$ .*

- (1) *Exponent four*:  $4 \cdot [\mathcal{M}_{p,g}] = 0$
- (2) *Idempotence*:  $2(V_2 - 1) \cdot [\mathcal{M}_{p,1}] = 0$
- (3) *Exponent two*:  $2 \cdot ([\mathcal{M}_{x,g}] - [\mathcal{M}_{1,xg}]) = 0$
- (4) *Nilpotence*:  $V_2 \cdot ([\mathcal{M}_{x,g}] - [\mathcal{M}_{1,xg}]) = 0$

Finally we conclude with a determination of the Verschiebung module extension  $\text{UNil}_3(\mathbb{Z}[C_2])$ , through the eyes of the Connolly–Ranicki isomorphism (2.2).

**Theorem 2.11.** *The extension of  $\mathcal{V}$ -modules in Theorem 2.4 is trivial.*

### 3. MAIN PROOFS USING RELATIONS

*Proof of Theorem 1.1.* Denote  $S$  as the Sylow 2-subgroup of  $F$ . Since  $S$  is normal and abelian, by the reduction isomorphism of [Kha, Theorem 1.1] and the Connolly–Ranicki isomorphism  $r$  of Theorem 2.2, it suffices to show that:

$$NL_n^h(\mathbb{Z}[S]) = 0 \quad \text{if } n \equiv 0, 1 \pmod{4}$$

and the following induced map is an isomorphism:

$$NL_2^h(\mathbb{Z}[S]) \longrightarrow NL_2^h(\mathbb{F}_2).$$

We induct on the order of  $S$ . If  $|S| = 1$ , then recall from Remark 2.3 that

$$NL_n(\mathbb{Z}[S]) = NL_n(\mathbb{Z}[1]) = 0 \quad \text{if } n \equiv 0, 1 \pmod{4}$$

and the following induced map is an isomorphism:

$$NL_2(\mathbb{Z}[S]) = NL_2(\mathbb{Z}[1]) \longrightarrow NL_2(\mathbb{F}_2).$$

Otherwise suppose  $|S| > 1$ . Since  $S$  has exponent two, there is a decomposition

$$S = S' \times C_2$$

as an internal direct product of groups of exponent two. Then the Mayer–Vietoris sequence of [Kha, Proposition 6.1] specializes to:

$$\cdots NL_{n+1}(\mathbb{F}_2) \xrightarrow{\partial} NL_n(\mathbb{Z}[S]) \longrightarrow \bigoplus_2 NL_n(\mathbb{Z}[S']) \longrightarrow NL_n(\mathbb{F}_2) \xrightarrow{\partial} \cdots$$

Observe, by inductive hypothesis and Remark 2.3, that  $NL_n(\mathbb{Z}[S']) = NL_n(\mathbb{F}_2) = 0$  for all  $n \equiv 0, 1 \pmod{4}$ . So we obtain  $NL_0(\mathbb{Z}[S]) = 0$  and an exact sequence

$$0 \longrightarrow NL_2(\mathbb{Z}[S]) \longrightarrow \bigoplus_2 NL_2(\mathbb{Z}[S']) \longrightarrow NL_2(\mathbb{F}_2) \xrightarrow{\partial} NL_1(\mathbb{Z}[S]) \longrightarrow 0.$$

But the following induced map is an isomorphism, by inductive hypothesis:

$$NL_2(\mathbb{Z}[S']) \longrightarrow NL_2(\mathbb{F}_2).$$

Therefore

$$NL_1(\mathbb{Z}[S]) = 0$$

and the following composite of induced maps is an isomorphism:

$$NL_2(\mathbb{Z}[S]) \longrightarrow NL_2(\mathbb{Z}[S']) \longrightarrow NL_2(\mathbb{F}_2).$$

This concludes the induction on  $|S|$ .  $\square$

*Proof of Theorem 2.4.* The exact sequence of [Kha, Proposition 6.1] becomes

$$NL_{n+1}(\mathbb{F}_2) \xrightarrow{\partial} NL_n(\mathbb{Z}[C_2]) \xrightarrow{\begin{pmatrix} i^- \\ i^+ \end{pmatrix}} NL_n(\mathbb{Z}) \oplus NL_n(\mathbb{Z}) \xrightarrow{(j^- - j^+)} NL_n(\mathbb{F}_2).$$

Since this sequence is functorial, it must consist of  $\mathcal{V}$ -module morphisms. It follows by Orientable Reduction [Kha, Prop. 4.1] that there is the commutative diagram of Figure 3.1 with top row exact, where  $\varepsilon : C_2 \rightarrow C_2$  is the trivial map and

$$\begin{array}{ccccccc} 0 & \longrightarrow & NL_4(\mathbb{F}_2) & \xrightarrow{\partial} & NL_3(\mathbb{Z}[C_2]) & \xrightarrow{\begin{pmatrix} i^- \\ i^+ \end{pmatrix}} & NL_3(\mathbb{Z}) \oplus NL_3(\mathbb{Z}) \longrightarrow 0 \\ & & \parallel & & \downarrow 1-\varepsilon & \searrow i^- - i^+ & \downarrow \text{proj}_{\text{skew-diag}} \\ 0 & \longrightarrow & NL_4(\mathbb{F}_2) & \xrightarrow{\tilde{\partial}} & \widetilde{NL}_3(\mathbb{Z}[C_2]) & \xrightarrow{i^-} & NL_3(\mathbb{Z}) \longrightarrow 0 \end{array}$$

FIGURE 3.1. Reduction of  $NL_3$

$$\tilde{\partial} := (\mathbf{1} - \varepsilon) \circ \partial.$$

The map  $\tilde{\partial}$  is a monomorphism, since  $i^+ \circ \partial = 0$  and the left square commutes. The map  $i^-$  is an epimorphism, since the projection  $\text{proj}_{\text{skew-diag}}$  onto the skew-diagonal is surjective and the right square commutes. Exactness at  $\widetilde{NL}_3(\mathbb{Z}[C_2])$  follows from its definition and exactness of the top row at  $NL_3(\mathbb{Z}[C_2])$ . Thus the bottom row exists and is an exact sequence of  $\mathcal{V}$ -modules.  $\square$

*Proof of Proposition 2.8(1).* According to [Ran81, pp. 517–519], the boundary map

$$\partial = \partial_{i^-} \circ \delta : L_4(\mathbb{F}_2[x]) \rightarrow L_3(\mathbb{Z}[C_2][x])$$

for our cartesian square is defined in general in terms of pullback modules by

$$(A'^r, \psi') \longmapsto \left( (B^r, \mathbf{1}, B'^r), \left( \begin{pmatrix} \mathbf{1} - (\chi + \chi^*) \circ \phi, \mathbf{0} \\ \phi, \mathbf{1} \end{pmatrix}, (\psi - \phi \circ \chi \circ \phi, \mathbf{0}) \right) (B^r, \phi', B'^r) \right).$$

It sends a Witt class of a rank  $r$  nonsingular form over  $A' = \mathbb{F}_2[x]$  to the Witt class of split formation over  $A = \mathbb{Z}[C_2][x]$  obtained by pullback of the boundary formation of the lifted form over  $B = \mathbb{Z}[x]$  and of the hyperbolic formation over  $B' = \mathbb{Z}[x]$ . The form  $\psi$  over  $B$  lifts the input form  $\psi'$  over  $A'$ . Their symmetrizations are denoted

$$\phi := \psi + \psi^* : B^r \longrightarrow (B^r)^* \quad \text{and} \quad \phi' := \psi' + \psi'^* : B'^r \longrightarrow (B'^r)^*.$$

The morphism  $\chi : (B'^r)^* \rightarrow B'^r$  lifts the map

$$\chi' := (\phi')^{-1} \circ \psi' \circ (\phi')^{-1} : (A'^r)^* \longrightarrow A'^r.$$

Now we compute these morphisms in our situation. Let  $p \in x\mathbb{Z}[x]$ . Recall (2.3) and take

$$(A'^r, \psi') = P_{q,1} = \left( \mathbb{F}_2[x]^2, \begin{bmatrix} q & 1 \\ 0 & 1 \end{bmatrix} \right).$$

Choose a lift

$$(B^r, \psi) = \left( \mathbb{Z}[x]^2, \begin{bmatrix} q & 1 \\ 0 & 1 \end{bmatrix} \right).$$

Then we obtain and select

$$\chi' = \begin{bmatrix} 1 & 0 \\ 1 & q \end{bmatrix} : \mathbb{F}_2[x]^2 \rightarrow \mathbb{F}_2[x]^2 \quad \chi = \begin{bmatrix} -1 & 0 \\ 1 & -q \end{bmatrix} : \mathbb{Z}[x]^2 \rightarrow \mathbb{Z}[x]^2.$$

Using the pullback module structure [Ran81, p. 507]

$$\mathbb{Z}[C_2][x] \xrightarrow{\cong} (\mathbb{Z}[x], \mathbf{1} : \mathbb{F}_2[x] \rightarrow \mathbb{F}_2[x], \mathbb{Z}[x]); \quad (m + nT) \mapsto (m - n, m + n),$$

the pullback formation is

$$\begin{aligned} \partial[P_{q,1}] &= \left( (\mathbb{Z}[x]^2, \mathbf{1}, \mathbb{Z}[x]^2), \left( \left( \begin{bmatrix} 4q & 0 \\ 0 & 4q \end{bmatrix}, \mathbf{0} \right), \left( \begin{bmatrix} 2q & 1 \\ 1 & 2 \end{bmatrix}, \mathbf{1} \right) \right), \left( \begin{bmatrix} 4q^2 & 4q \\ 0 & 4q \end{bmatrix}, \mathbf{0} \right) (\mathbb{Z}[x]^2, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \mathbb{Z}[x]^2) \right) \\ &= \left( (\mathbb{Z}[x]^2, \mathbf{1}, \mathbb{Z}[x]^2), \left( \left( \begin{bmatrix} 0 & 4q \\ 4q & 0 \end{bmatrix}, \mathbf{0} \right), \left( \begin{bmatrix} 1 & 2q \\ 2 & 1 \end{bmatrix}, \mathbf{1} \right) \right), \left( \begin{bmatrix} 4q & 0 \\ 4q & 4q^2 \end{bmatrix}, \mathbf{0} \right) (\mathbb{Z}[x]^2, \mathbf{1}, \mathbb{Z}[x]^2) \right) \\ &= \left( \mathbb{Z}[C_2][x]^2, \left( \begin{bmatrix} 0 & 2(1-T)q \\ 2(1-T)q & 0 \\ 1 & (1-T)q \\ (1-T) & 1 \end{bmatrix}, \begin{bmatrix} 2(1-T)q & 0 \\ 2(1-T)q & 2(1-T)q^2 \end{bmatrix} \right) \mathbb{Z}[C_2][x]^2 \right) \\ &= [\mathcal{Q}_q]. \end{aligned}$$

□

*Proof of Proposition 2.8(2).* Clearly  $i^-(\mathcal{M}_{p,g}) = \mathcal{N}_{p,g}$ . Note that the second lagrangian  $G$  of  $i^+(\mathcal{M}_{p,g})$  is

$$\text{Im} \begin{pmatrix} p & 1 \\ 1 & 0 \\ 2 & 0 \\ 0 & 2 \end{pmatrix} = \text{Im} \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 0 \\ -2p & 2 \end{pmatrix} = \text{Im} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \\ 2 & -2p \end{pmatrix}.$$

Therefore  $i^+(\mathcal{M}_{p,g})$  is a graph formation over  $\mathbb{Z}[x]$ , hence represents 0 in  $NL_3(\mathbb{Z})$ . □

*Proof of Corollary 2.10(1).* Note, by Proposition 2.9(1) and the relations (2.3) in  $NL_4(\mathbb{F}_2)$ , that

$$4 \cdot [\mathcal{M}_{p,g}] = 2 \cdot [\mathcal{Q}_{pg}] + 2 \cdot [\mathcal{M}_{2p,g}] = 2 \cdot [\mathcal{Q}_{pg}] + [\mathcal{Q}_{2pg}] + [\mathcal{M}_{4p,g}] = [\mathcal{M}_{4p,g}].$$

There is an isomorphism

$$(\mathbf{1}, \mathbf{1}, \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}) : \mathcal{M}_{0,g} \longrightarrow \mathcal{M}_{4p,g}$$

of split  $(-1)$ -quadratic formations over  $\mathbb{Z}[C_2][x]$ , see [Ran81, p. 69, defn.]. Therefore, as cobordism classes in  $NL_3(\mathbb{Z}[C_2])$ , we obtain

$$4 \cdot [\mathcal{M}_{p,g}] = [\mathcal{M}_{4p,g}] = [\mathcal{M}_{0,g}] = 0,$$

by [Ran81, Proposition 1.6.4] and since  $\mathcal{M}_{0,g}$  is a graph formation.  $\square$

*Proof of Corollary 2.10(2).* Note, by Proposition 2.9(1) and the relations (2.3) in  $NL_4(\mathbb{F}_2)$ , that

$$\begin{aligned} 2(V_2 - 1) \cdot [\mathcal{M}_{p,1}] &= (V_2 - 1) \cdot ([\mathcal{Q}_p] + [\mathcal{M}_{2p,1}]) \\ &= [\mathcal{M}_{2V_2p,1}] - [\mathcal{M}_{2p,1}] \\ &= \sum_{k=1}^n p_k \cdot ([\mathcal{M}_{2(x^k)^2,1}] - [\mathcal{M}_{2(x^k),1}]) \end{aligned}$$

where we write the polynomial

$$p = p_1x + \cdots + p_nx^n \in \text{Ker}(\text{aug}_0)$$

for some  $n \in \mathbb{N}$  and  $p_1, \dots, p_n \in \mathbb{Z}$ . But by Proposition 2.9(4), we have

$$[\mathcal{M}_{2(x^k)^2,1}] - [\mathcal{M}_{2(x^k),1}] = 0 \quad \text{for all } k > 0.$$

Therefore

$$2(V_2 - 1) \cdot [\mathcal{M}_{p,1}] = 0. \quad \square$$

*Proof of Corollary 2.10(3).* Note by Proposition 2.9(1,2) and the relations (2.3) in  $NL_4(\mathbb{F}_2)$  that

$$\begin{aligned} 2 \cdot ([\mathcal{M}_{x,g}] - [\mathcal{M}_{1,xg}]) &= ([\mathcal{Q}_{xg}] - [\mathcal{Q}_{xg}]) + ([\mathcal{M}_{2x,g}] - [\mathcal{M}_{2x,g}]) \\ &= [\mathcal{M}_{2g,x}] - [\mathcal{M}_{2xg,1}]. \end{aligned}$$

By Proposition 2.9(1) using the fact that  $[\mathcal{Q}_q] = 0$  if  $q$  is a multiple of 2, and since  $g$  has  $\mathbb{Z}$ -coefficients, we may assume that  $g = x^k$  for some  $k \in \mathbb{N}$  in order to show that the right-hand term vanishes. If  $k = 2i$  is even, then by Proposition 2.9(3,2), note

$$[\mathcal{M}_{2g,x}] = [\mathcal{M}_{2(x^i)^2,x}] = [\mathcal{M}_{2,(x^i)^2x}] = [\mathcal{M}_{2,xg}] = [\mathcal{M}_{2xg,1}].$$

Otherwise suppose  $k = 2i + 1$  is odd. Then by Proposition 2.9(4) twice and by induction on  $k$ , note

$$[\mathcal{M}_{2g,x}] = [\mathcal{M}_{2(x^i)^2x,x}] = [\mathcal{M}_{2(x^i),x}] = [\mathcal{M}_{2(x^{i+1}),1}] = [\mathcal{M}_{2(x^{i+1})^2,1}] = [\mathcal{M}_{2xg,1}].$$

Therefore for all  $g \in \mathbb{Z}[x]$  we obtain

$$2 \cdot ([\mathcal{M}_{x,g}] - [\mathcal{M}_{1,xg}]) = 0. \quad \square$$

*Proof of Corollary 2.10(4).* Note by Proposition 2.9(3) that

$$V_2 \cdot ([\mathcal{M}_{x,g}] - [\mathcal{M}_{1,xg}]) = [\mathcal{M}_{x^2,V_2g}] - [\mathcal{M}_{1,x^2V_2g}] = 0. \quad \square$$



*Proof of Theorem 2.11.* Consider the  $\mathcal{V}$ -module morphism

$$s : NL_3(\mathbb{Z}) \longrightarrow \widetilde{NL}_3(\mathbb{Z}[C_2])$$

given additively by

$$\begin{aligned} V_n \cdot [\mathcal{N}_{x,1}] &\longmapsto V_n \cdot [\mathcal{M}_{x,1}] \\ V_n \cdot ([\mathcal{N}_{x,x^p}] - [\mathcal{N}_{1,xx^p}]) &\longmapsto V_n \cdot ([\mathcal{M}_{x,x^p}] - [\mathcal{M}_{1,xx^p}]). \end{aligned}$$

The section  $s$  is a well-defined  $\mathcal{V}$ -module morphism by Corollary 2.10.  $\square$

#### 4. SOME ALGEBRAIC SURGERY MACHINES

The remaining proofs of all parts of Proposition 2.9 are technical—algebraic surgery and gluing are required. The first machine has input certain quadratic formations and has output quadratic forms.

**Lemma 4.1.** *Suppose  $(C, \psi)$  is a 1-dimensional  $(-1)$ -quadratic Poincaré complex over  $\mathbb{Z}[C_2][x]$  satisfying the following hypotheses.*

- (a) *The 1-dimensional chain complex  $C$  over  $\mathbb{Z}[C_2][x]$  has modules  $C_1 = C_0$  and differential  $d_C = 2 \cdot \mathbf{1}$ .*
- (b) *There is a null-cobordism*

$$(f : i^-(C) \rightarrow D, (\delta\psi, i^-(\psi)) \in W_{\%}(f, -1)_2)$$

*over  $\mathbb{Z}[x]$  such that  $f_0 = \mathbf{1} : C_0 \rightarrow D_0$  and  $\delta\psi_2 = 0 : D^0 \rightarrow D_0$ .*

- (c) *The quadratic Poincaré complex  $i^+(C, \psi)$  over  $\mathbb{Z}[x]$  corresponds to a graph formation.*

*Then we obtain the following conclusions.*

- (1) *There exists a 2-dimensional  $(-1)$ -quadratic Poincaré complex  $(F, \Psi)$  over  $\mathbb{F}_2[x]$  such that*

$$\partial[\overline{S}(F, \Psi)] = [\overline{S}(C, \psi)].$$

*Here,*

$$\partial : L_4(\mathbb{F}_2[x]) \longrightarrow L_3(\mathbb{Z}[C_2][x])$$

*is the boundary map of the Mayer–Vietoris sequence of Rim’s cartesian square, and  $\overline{S}$  is the skew-suspension isomorphism.*

- (2) *The instant surgery obstruction  $\Omega(F, \Psi)$  is Witt equivalent to the nonsingular  $(+1)$ -quadratic form  $j^-(D^1, \delta\psi_0)$  over  $\mathbb{F}_2[x]$ .*

The next machine constructs inputs for the above one given a lagrangian of a certain linking form. It is obtained as a specialization of [Ran81, Proof 3.4.5(ii)]<sup>2</sup>.

**Lemma 4.2.** *Suppose  $(C, \psi)$  is a 1-dimensional  $(-1)$ -quadratic Poincaré complex over  $\mathbb{Z}[C_2][x]$  satisfying the following hypotheses.*

- (a) *The 1-dimensional chain complex  $C$  over  $\mathbb{Z}[C_2][x]$  has modules  $C_1 = C_0$  and differential  $d_C = 2 \cdot \mathbf{1}$ .*
- (b) *There exists a lagrangian  $L$  of the nonsingular  $(+1)$ -quadratic linking form  $(N, b, q)$  over  $(\mathbb{Z}[x], (2)^\infty)$  associated to  $i^-(C, \psi)$ .*
- (c) *The evaluation  $i^+(C, \psi)$  corresponds to a graph formation over  $\mathbb{Z}[x]$ .*

*Choose a finitely generated projective module  $P$  over  $\mathbb{Z}[C_2][x]$  and morphisms*

<sup>2</sup>See errata for the formulas at <http://www.maths.ed.ac.uk/~aar/books/exacterr.pdf>.

(i)  $\pi : P \rightarrow C^1$  monic with image

$$i^-(\pi)(P) = e^{-1}(L),$$

where the quotient map  $e$  is

$$e : i^-(C^1) \longrightarrow N := \text{Cok}(i^-(d_C^* : C^0 \rightarrow C^1))$$

(ii)  $\chi : P \rightarrow P^*$  satisfying the de-symmetrization identity

$$(\pi^{-1} \circ d_C^*)^* \circ (\chi + \chi^*) = (\tilde{\psi}_0 - \psi_0^*) \circ \pi : P \rightarrow C_0.$$

Then we obtain the following conclusions.

(1) Define a quadratic cycle  $\hat{\psi} \in W_{\%}(C, -1)_1$  by

$$\begin{aligned} \hat{\psi}_0 &:= \psi_0 : C^0 \longrightarrow C_1 & \tilde{\psi}_0 &:= \tilde{\psi}_0 : C^1 \longrightarrow C_0 \\ \hat{\psi}_1 &:= (\pi^{-1} \circ d_C^*)^* \circ \chi \circ (\pi^{-1} \circ d_C^*) - \tilde{\psi}_0 \circ d_C^* : C^0 \longrightarrow C_0. \end{aligned}$$

Then the quadratic cycle  $\hat{\psi}$  is homologous to  $\psi$  in  $W_{\%}(C, -1)_1$  over  $\mathbb{Z}[C_2][x]$ .

(2) Define a chain complex  $D = \left\{ D_1 \xrightarrow{d_D} D_0 \right\}$  with modules

$$D_1 := i^-(P^*) \quad \text{and} \quad D_0 := i^-(C_0)$$

and with differential

$$d_D := i^-(\pi^{-1} \circ d_C^*)^*.$$

Define a chain map  $f : i^-(C) \rightarrow D$  by

$$f_0 := \mathbf{1} : i^-(C_0) \longrightarrow D_0 \quad f_1 := i^-(\pi^*) : i^-(C_1) \longrightarrow D_1.$$

Define a quadratic chain  $\delta\psi \in W_{\%}(D, -1)_2$  by

$$\begin{aligned} \delta\psi_0 &:= -i^-(\chi)^* : D^1 \longrightarrow D_1 & \delta\psi_1 &:= -i^-(\chi) \circ d_D^* : D^0 \longrightarrow D_1 \\ \tilde{\delta\psi}_1 &:= \tilde{\psi}_0 \circ i^-(\pi) : D^1 \longrightarrow D_0 & \delta\psi_2 &:= 0 : D^0 \longrightarrow D_0. \end{aligned}$$

Then

$$\left( f : i^-(C) \rightarrow D, (\delta\psi, i^-(\hat{\psi})) \in W_{\%}(f, -1)_2 \right)$$

is a null-cobordism over  $\mathbb{Z}[x]$ .

(3) The evaluation  $i^+(C, \hat{\psi})$  corresponds to a graph formation over  $\mathbb{Z}[x]$ .

Composition of the lemmas yields immediately the following result.

**Proposition 4.3.** *Suppose  $(C, \psi)$  satisfies Hypotheses (a,b,c) of Lemma 4.2, and choose  $P, \pi, \chi$  accordingly. Then there exists a 2-dimensional  $(-1)$ -quadratic complex  $(F, \Psi)$  over  $\mathbb{F}_2[x]$  such that*

$$\partial[(F, \Psi)] = [(C, \hat{\psi})] = [(C, \psi)]$$

as cobordism classes in

$$L_1(\mathbb{Z}[C_2][x], -1) \xrightarrow{\overline{S} \cong} L_3(\mathbb{Z}[C_2][x]),$$

and that its instant surgery obstruction is

$$[i(F, \Psi)] = [j^-(D^1, \delta\psi_0)] = [k(P, -\chi^*)]$$

as Witt classes of the nonsingular  $(+1)$ -quadratic forms over  $\mathbb{F}_2[x]$ .  $\square$

Now we show why these machines work.

*Proof of Lemma 4.1.* We shall put together the information in the hypotheses using a technique called “algebraic gluing” [Ran81, §1.7]. The resultant object  $(F, \Psi)$  is a union [Ran81, pp. 77–78] over  $\mathbb{F}_2[x]$ . The more efficient “direct union” [Ran81, pp. 79–80] does not apply here since the null-cobordisms  $(D, \delta\psi)$  and  $(E, 0)$  are non-split in general.

First, define a chain complex  $E = \{E_1 \rightarrow 0\}$  over  $\mathbb{Z}[x]$  with module

$$E_1 := i^+(C_1),$$

and a chain map  $g : i^+(C) \rightarrow E$  by

$$g_1 := \mathbf{1} : i^+(C_1) \rightarrow E_1.$$

Then the quadratic pair

$$(g : i^+(C) \rightarrow E, (0, i^+(\psi) \in W_{\%}(g, -1)_2))$$

is the data for an algebraic surgery. Consider the 2-dimensional mapping cone

$$\mathcal{C}(g) = \left( i^+(C_1) \xrightarrow{\begin{pmatrix} -1 \\ 2\mathbf{1} \end{pmatrix}} E_1 \oplus i^+(C_0) \longrightarrow 0 \right).$$

Note that

$$H^2(E) = H_0(\mathcal{C}(g)) = 0 \quad \text{and} \quad H^0(E) = H_2(\mathcal{C}(g)) = 0.$$

Observe that

$$H^1(E) = E^1 \quad \text{and} \quad \text{proj}_* : H_1(\mathcal{C}(g)) \xrightarrow{\cong} i^+(C_0).$$

Then the homological Poincaré duality map  $H^1(E) \rightarrow H_1(\mathcal{C}(g))$  is given by

$$i^+(\tilde{\psi}_0 - \psi_0^*) : E^1 \longrightarrow i^+(C_0).$$

By hypothesis,  $i^+(C, \psi)$  represents a graph formation

$$\left( F, \left( \begin{pmatrix} \gamma \\ \mu \end{pmatrix}, \theta \right) G \right),$$

which means that  $\gamma : G \rightarrow F$  an isomorphism. According to [Ran80, Proof 2.5], the representation is given by

$$F = i^+(C_1) \quad \text{and} \quad G = i^+(C^0)$$

$$\gamma = i^+(\tilde{\psi}_0^* - \psi_0) \quad \text{and} \quad \mu = i^+(d_C^*) \quad \text{and} \quad \theta = -i^+(\psi + d_C \circ \psi_0).$$

Thus the map  $H^1(E) \rightarrow H_1(\mathcal{C}(g))$  is given by the isomorphism  $\gamma^*$ . Since the Poincaré duality map  $E^{2-*} \rightarrow \mathcal{C}(g)$  of projective module chain complexes induces isomorphisms in homology, it must be a chain homotopy equivalence. Thus the following 2-dimensional  $(-1)$ -quadratic pair is Poincaré:

$$(g : i^+(C) \rightarrow E, (0, i^+(\psi))).$$

Next, define a 2-dimensional  $(-1)$ -quadratic Poincaré complex  $(F, \Psi)$  over  $\mathbb{F}_2[x]$  as the union (see [Ran81, pp. 77–78])

$$(F, \Psi) := j^- \left( f : i^-(C) \rightarrow D, (-\delta\psi, -i^-(\psi)) \right) \bigcup_{k(C, \psi)} j^+ \left( g : i^+(C) \rightarrow E, (0, i^+(\psi)) \right),$$

where  $k$  is composite morphism of rings with involution:

$$k := j^- \circ i^- = j^+ \circ i^+ : \mathbb{Z}[C_2] \longrightarrow \mathbb{F}_2.$$

By construction,

$$\partial[(F, \Psi)] = [(C, \psi)],$$

where the boundary map

$$\partial : L_4(\mathbb{F}_2[x]) \longrightarrow L_3(\mathbb{Z}[C_2][x])$$

is defined in [Ran81, Props. 6.3.1, 6.1.3] for our cartesian square. For simplicity, we suppress the morphisms  $i^\pm, j^\pm, k$  in the remainder of the proof.

The 2-dimensional chain complex  $F$  over  $\mathbb{F}_2[x]$  has modules

$$F_2 = C_1 \quad F_1 = D_1 \oplus C_0 \oplus E_1 \quad F_0 = D_0$$

and differentials

$$d_F^2 = \begin{pmatrix} -f_1 \\ d_C \\ -g_1 \end{pmatrix} : F_2 \rightarrow F_1 \quad d_F^1 = \begin{pmatrix} d_D & f_0 & 0 \end{pmatrix} : F_1 \rightarrow F_0.$$

The quadratic cycle  $\Psi \in W_{\%}(F, -1)_2$  has components

$$\begin{aligned} \Psi_0^2 &= (-\psi_0 \circ f_0^*) : F^0 \rightarrow F_2 & \Psi_0^1 &= \begin{pmatrix} -\delta\psi_0 & 0 & 0 \\ \tilde{\psi}_0 \circ f_1^* & \psi_1^* & 0 \\ 0 & g_1 \circ \psi_0 & 0 \end{pmatrix} : F^1 \rightarrow F_1 \\ \Psi_0^0 &= (0) : F^2 \rightarrow F_0 & \Psi_1^1 &= \begin{pmatrix} -\delta\psi_1 \\ \psi_1 \circ f_0^* \\ 0 \end{pmatrix} : F^0 \rightarrow F_1 \\ \Psi_1^0 &= \begin{pmatrix} -\delta\tilde{\psi}_1 & 0 & 0 \end{pmatrix} : F^1 \rightarrow F_0 & \Psi_2^0 &= (0) : F^0 \rightarrow F_0. \end{aligned}$$

The differential

$$(d_F^1)^* : F^0 \longrightarrow F^1$$

is a split monomorphism, since  $f_0 = \mathbf{1} : C_0 \rightarrow D_0$ . Hence the instant surgery obstruction [Ran80, Prop. 4.3] is represented by

$$\Omega(F, \Psi) = \left( D^1 \oplus E^1 \oplus C_1, \begin{pmatrix} \delta\psi_0 & 0 & -f_1 \\ 0 & 0 & -\mathbf{1} \\ 0 & 0 & 0 \end{pmatrix} \right).$$

This is Witt equivalent to the (necessarily) nonsingular (+1)-quadratic form  $(D^1, \delta\psi_0)$  over  $\mathbb{F}_2[x]$ .  $\square$

*Proof of Lemma 4.2.* Indeed  $\hat{\psi} \in W_{\%}(C, -1)_1$  is a quadratic cycle, since

$$\begin{aligned} \hat{\psi}_1 + \hat{\psi}_1^* &= (\pi^{-1} \circ d_C^*)^* \circ (\chi + \chi^*) \circ (\pi^{-1} \circ d_C^*) - (\tilde{\psi}_0 \circ d_C^* + d_C \circ \tilde{\psi}_0^*) \\ &= (\tilde{\psi}_0 - \psi_0^*) \circ d_C^* - \tilde{\psi}_0 \circ d_C^* - d_C \circ \tilde{\psi}_0^* \\ &= -(d_C \circ \psi_0 + \tilde{\psi}_0 \circ d_C^*)^* \\ &= (\psi_1 + \psi_1^*)^* \\ &= \psi_1 + \psi_1^*. \end{aligned}$$

A similar check shows that  $f : i^-(C) \rightarrow D$  is a chain map and that

$$(f : i^-(C) \rightarrow D, (\delta\psi, i^-(\hat{\psi})))$$

is a 2-dimensional  $(-1)$ -quadratic pair over  $\mathbb{Z}[x]$ . It is Poincaré (see [Ran81, p. 259]), since it is the data for an algebraic surgery to a contractible complex, killing the lift  $i^-(P)$  of the lagrangian  $L$ .

The quadratic cycles  $\widehat{\psi}$  and  $\psi$  are homologous<sup>3</sup>: the differences  $\widehat{\psi}_0 - \psi_0$  and  $\widetilde{\psi}_0 - \widetilde{\psi}_0$  are zero, and the difference  $\widehat{\psi}_1 - \psi_1$  is  $(-1)$ -symmetric (see above calculation). Therefore, the latter difference is  $(-1)$ -even since  $\widehat{H}^0(\mathbb{Z}[x], -1) = 0$ . Finally,  $i^+(C, \widehat{\psi})$  corresponds to the same graph formation as  $i^+(C, \psi)$ , except that their Hessians have difference  $\widehat{\theta} - \theta = \psi_1 - \widehat{\psi}_1$ .  $\square$

## 5. REMAINING PROOFS OF RELATIONS

Using our machine (4.3), we grind out the primary relations (2.9) in  $\widetilde{NL}_3(\mathbb{Z}[C_2])$  as a  $\mathcal{V}$ -module.

*Proof of Proposition 2.9(1).* Let  $(C, \psi)$  be a 1-dimensional  $(-1)$ -quadratic Poincaré complex associated to the following nonsingular split  $(-1)$ -quadratic formation over  $\mathbb{Z}[C_2][x]$ :

$$\mathcal{M}_{p_1, g} \oplus \mathcal{M}_{p_2, g} \oplus \mathcal{M}_{p_1 + p_2, g}.$$

In particular, it has modules  $C_1 = C_0$  of rank 6 and differential  $d_C = 2 \cdot \mathbf{1}$ . Consider the exponent two linking form  $(N, b, q)$  over  $(\mathbb{Z}[x], (2)^\infty)$  associated to the evaluation  $i^-(C, \psi)$ , defined as

$$(N, b, q) = \mathcal{N}_{p_1, g} \oplus \mathcal{N}_{p_2, g} \oplus -\mathcal{N}_{p_1 + p_2, g}.$$

Define a lift  $\pi : P \rightarrow C^1$  of a lagrangian  $L$  of  $(N, b, q)$  and a morphism  $\chi : P \rightarrow P^*$  by

$$\pi := \begin{bmatrix} 0 & 1 & 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad \chi := \begin{bmatrix} g & 1 & g & 1 & 2g & 1 \\ 0 & 0 & 0 & p_1 & 1 & p_2 \\ 0 & 0 & 0 & 1 & 2g & 0 \\ 0 & 0 & 0 & p_1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2g & 0 \\ 0 & 0 & 0 & 0 & 0 & p_2 \end{bmatrix}.$$

It is straightforward to verify that  $i^-(\pi)(P)$  is the inverse image of a lagrangian  $L$  and that  $\chi$  satisfies the de-symmetrization identity. Therefore, by Proposition 4.3, we obtain a 2-dimensional  $(-1)$ -quadratic Poincaré complex  $(F, \Psi)$  over  $\mathbb{F}_2[x]$  such that

$$\partial[(F, \Psi)] = [(C, \widehat{\psi})] = [(C, \psi)] \quad \text{and} \quad [\Omega(F, \Psi)] = [k(P, -\chi^*)].$$

In classical notation, we have that  $(F, \Psi)$  is represented by the nonsingular  $(+1)$ -quadratic form

$$(M, \lambda, \mu) := \left( \bigoplus_6 \mathbb{F}_2[x], \begin{bmatrix} 0 & 1 & g & 1 & 0 & 1 \\ 1 & 0 & 0 & p_1 & 1 & p_2 \\ g & 0 & 0 & 1 & 0 & 0 \\ 1 & p_1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & p_2 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} g \\ 0 \\ 0 \\ p_1 \\ 0 \\ p_2 \end{bmatrix} \right).$$

<sup>3</sup>In general,  $\widehat{\psi}$  and  $\psi$  are *quadratic* homotopy equivalent [Ran81, p. 71 Defn., Prop. 3.4.5(ii)].

Its pullback along the choice (see [Wal99, Proof 5.3]) of automorphism

$$\alpha := \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & g & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & g & 0 \\ 0 & 0 & 0 & p_1 & 1 + p_1 g & p_2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} : M \longrightarrow M$$

is the symplectic form

$$\alpha^*(M, \lambda, \mu) = \left( \bigoplus_6 \mathbb{F}_2[x], \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} g \\ 0 \\ 0 \\ p_1 \\ p_1 g^2 \\ p_2 \end{bmatrix} \right)$$

which has Arf invariant  $[q]$ , where  $q := (p_1 g)(p_2 g)$ . So, by Remark 2.3, as cobordism classes in  $L_4(\mathbb{F}_2[x])$ , we must have  $[(F, \Psi)] = [P_{q,1}]$ . Therefore, by Proposition 2.8(1), we obtain

$$[\mathcal{M}_{p_1, g}] + [\mathcal{M}_{p_2, g}] - [\mathcal{M}_{p_1 + p_2, g}] = [(C, \psi)] = \partial[(F, \Psi)] = \partial[P_{q,1}] = \tilde{\partial}[P_{q,1}] = [\mathcal{Q}_q].$$

□

*Proof of Proposition 2.9(2).* Let  $(C, \psi)$  be a 1-dimensional  $(-1)$ -quadratic Poincaré complex associated to the following nonsingular split  $(-1)$ -quadratic formation over  $\mathbb{Z}[C_2][x]$ :

$$\mathcal{M}_{2p, g} \oplus -\mathcal{M}_{2g, p}.$$

In particular, it has modules  $C_1 = C_0$  of rank 4 and differential  $d_C = 2 \cdot \mathbf{1}$ . Consider the exponent two linking form  $(N, b, q)$  over  $(\mathbb{Z}[x], (2)^\infty)$  associated to evaluation  $i^-(C, \psi)$ , defined as

$$(N, b, q) = \mathcal{N}_{2p, g} \oplus -\mathcal{N}_{2g, p}.$$

Define a lift  $\pi : P \rightarrow C^1$  of a lagrangian  $L$  of  $(N, b, q)$  and a morphism  $\chi : P \rightarrow P^*$  by

$$\pi := \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \chi := \begin{bmatrix} 0 & 0 & 2p & 1 \\ 0 & 0 & 1 & 2g \\ 0 & 0 & 2p & 2 \\ 0 & 0 & 0 & 2g \end{bmatrix}.$$

The remainder follows by an argument similar to Proof 2.9(1). □

*Proof of Proposition 2.9(3).* Let  $(C, \psi)$  be a 1-dimensional  $(-1)$ -quadratic Poincaré complex associated to the following nonsingular split  $(-1)$ -quadratic formation over  $\mathbb{Z}[C_2][x]$ :

$$\mathcal{M}_{x^2 p, g} \oplus -\mathcal{M}_{p, x^2 g}.$$

In particular, it has modules  $C_1 = C_0$  of rank 4 and differential  $d_C = 2 \cdot \mathbf{1}$ . Consider the exponent two linking form  $(N, b, q)$  over  $(\mathbb{Z}[x], (2)^\infty)$  associated to the evaluation  $i^-(C, \psi)$ , defined as

$$(N, b, q) = \mathcal{N}_{x^2 p, g} \oplus -\mathcal{N}_{p, x^2 g}.$$

Define a lift  $\pi : P \rightarrow C^1$  of a lagrangian  $L$  of  $(N, b, q)$  and a morphism  $\chi : P \rightarrow P^*$  by

$$\pi := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x & 0 & 2 \\ x & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \chi := \begin{bmatrix} 0 & 0 & -xp & 1 \\ 0 & 0 & -1 & 2xg \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & 2g \end{bmatrix}.$$

The remainder follows by an argument similar to Proof 2.9(1).  $\square$

*Proof of Proposition 2.9(4).* Let  $(C, \psi)$  be a 1-dimensional  $(-1)$ -quadratic Poincaré complex associated to the following nonsingular split  $(-1)$ -quadratic formation over  $\mathbb{Z}[C_2][x]$ :

$$\mathcal{M}_{2p^2g,g} \oplus -\mathcal{M}_{2p,g}.$$

In particular, it has modules  $C_1 = C_0$  of rank 4 and differential  $d_C = 2 \cdot \mathbf{1}$ . Consider the exponent two linking form  $(N, b, q)$  over  $(\mathbb{Z}[x], (2)^\infty)$  associated to the evaluation  $i^-(C, \psi)$ , defined as

$$(N, b, q) = \mathcal{N}_{2p^2g,g} \oplus -\mathcal{N}_{2p,g}.$$

Define a lift  $\pi : P \rightarrow C^1$  of a lagrangian  $L$  of  $(N, b, q)$  and a morphism  $\chi : P \rightarrow P^*$  by

$$\pi := \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & p & 2 & 0 \\ 0 & 1 & 0 & 0 \\ p & 0 & 0 & 2 \end{bmatrix}, \quad \chi := \begin{bmatrix} 0 & p^2g & 1 & -2pg \\ 0 & p^2g & 1 + 2pg & -1 \\ 0 & 0 & 2g & 0 \\ 0 & 0 & 0 & -2g \end{bmatrix}.$$

The remainder follows by an argument similar to Proof 2.9(1).  $\square$

This concludes the calculation of  $\text{UNil}_*(\mathbb{Z}[C_2])$  as a Verschiebung module.

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